

# From one Reeb orbit to two

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## Abstract

We show that every (possibly degenerate) contact form on the three-sphere giving the tight contact structure has at least two embedded Reeb orbits. The same holds for any closed contact three-manifold satisfying a weak version of the “volume conjecture” in embedded contact homology. More generally, the weak volume conjecture implies that if there are only finitely many embedded Reeb orbits, then their symplectic actions are not all integer multiples of a single real number. The volume conjecture itself, which is expected to hold for every closed contact three-manifold, implies that either there are at least three embedded Reeb orbits, or there are two embedded Reeb orbits with an explicit upper bound on the product of their symplectic actions.

## 1 Statement of results

Let  $Y$  be a closed oriented 3-manifold. Recall that a *contact form* on  $Y$  is a 1-form  $\lambda$  on  $Y$  such that  $\lambda \wedge d\lambda > 0$ . A contact form  $\lambda$  determines a *contact structure*  $\xi := \text{Ker}(\lambda)$ , and the *Reeb vector field*  $R$  characterized by  $d\lambda(R, \cdot) = 0$  and  $\lambda(R) = 1$ . A *Reeb orbit* is a closed orbit of the vector field  $R$ , i.e. a map  $\gamma : \mathbb{R}/T\mathbb{Z} \rightarrow Y$  for some  $T > 0$  such that  $\gamma'(t) = R(\gamma(t))$ , modulo reparametrization. The Reeb orbit  $\gamma$  is *nondegenerate* if the linearized Reeb flow along  $\gamma$  does not have 1 as an eigenvalue, and the contact form  $\lambda$  is called nondegenerate if all Reeb orbits are nondegenerate.

The three-dimensional Weinstein conjecture, proved by Taubes [17], asserts that any contact form on a closed 3-manifold has at least one Reeb orbit. It is interesting to try to improve the lower bound on the number of Reeb orbits. In fact, it seems that the only known examples of contact forms on closed three-manifolds with only finitely many embedded Reeb orbits are certain contact forms on lens spaces with exactly two embedded Reeb orbits. (Here we consider  $S^3$  to be a lens space.) It is shown in [12, Thm. 1.3] that any nondegenerate contact form on a closed three-manifold  $Y$  has at least two embedded Reeb orbits; and if  $Y$  is not a lens space, then there are at least three embedded Reeb orbits. The main theorem of the present paper asserts that assuming a certain conjecture, one can prove the existence of at least two embedded Reeb orbits without the nondegeneracy assumption:

**Theorem 1.1.** *Let  $(Y, \xi)$  be a closed contact three-manifold satisfying the Weak Volume Conjecture, stated below. Then every (possibly degenerate) contact form  $\lambda$  on  $Y$  with  $\text{Ker}(\lambda) = \xi$  has at least two embedded Reeb orbits.*

As explained in Remarks 2.1 and 2.2 below, the Weak Volume Conjecture is known to hold for the tight contact structure on  $S^3$ , so we obtain:

**Corollary 1.2.** *Let  $\lambda$  be any contact form giving the tight contact structure on  $S^3$ . Then  $\lambda$  has at least two embedded Reeb orbits.*

Corollary 1.2 also has the following implication for Hamiltonian dynamics. Recall that if  $Y$  is a hypersurface in a symplectic manifold  $(X, \omega)$ , then the *characteristic foliation* on  $Y$  is the rank one foliation  $L_Y := \text{Ker}(\omega|_{TY})$ , and a *closed characteristic* in  $Y$  is an embedded loop in  $Y$  tangent to  $L_Y$ . If  $Y$  is a regular level set of a smooth function  $H : X \rightarrow \mathbb{R}$ , then closed characteristics on  $Y$  are the same as unparametrized embedded closed orbits of the Hamiltonian vector field  $X_H$  on  $Y$ . Now consider  $X = \mathbb{R}^4$  with the standard symplectic form  $\omega = \sum_{i=1}^2 dx_i dy_i$ . If  $Y$  is a star-shaped hypersurface in  $\mathbb{R}^4$ , meaning that it is transverse to the radial vector field, then

$$\lambda = \frac{1}{2} \sum (x_i dy_i - y_i dx_i)$$

restricts to a contact form on  $Y$  (giving the tight contact structure), and the unparametrized embedded Reeb orbits are the same as the closed characteristics. Thus Corollary 1.2 implies the following:

**Corollary 1.3.** *Any smooth compact star-shaped hypersurface in  $\mathbb{R}^4$  has at least two closed characteristics.*

There are a number of previous results related to Corollary 1.3. Hofer-Wysocki-Zehnder showed in [4, Thm. 1.1] that any strictly convex hypersurface in  $\mathbb{R}^4$  has either 2 or infinitely many closed characteristics, and in [5, Cor. 1.10] that any nondegenerate contact form on  $S^3$  giving the tight contact structure has either two or infinitely many embedded Reeb orbits, provided that all stable and unstable manifolds of the hyperbolic periodic orbits intersect transversally. More recently, Long [15] has shown that any symmetric, compact star-shaped hypersurface in  $\mathbb{R}^4$  has at least two closed characteristics. And in higher dimensions, Wang [23] has shown that there are at least  $\lfloor \frac{n+1}{2} \rfloor + 1$  closed characteristics on every compact strictly convex hypersurface  $\Sigma$  in  $\mathbb{R}^{2n}$ . It has long been conjectured that there are at least  $n$  closed characteristics on every compact convex hypersurface in  $\mathbb{R}^{2n}$ ; for example, almost the same conjecture appears in [2, Conj. 1].

The method used to prove Theorem 1.1 yields a slightly more general result. To state it, if  $\gamma$  is a Reeb orbit, define its *symplectic action* by

$$\mathcal{A}(\gamma) := \int_{\gamma} \lambda.$$

We then have:

**Theorem 1.4.** *Let  $(Y, \xi)$  be a closed contact three-manifold satisfying the Weak Volume Conjecture. Let  $\lambda$  be a contact form with  $\text{Ker}(\lambda) = \xi$  having only finitely many embedded Reeb orbits  $\gamma_1, \dots, \gamma_m$ . Then their symplectic actions  $\mathcal{A}(\gamma_1), \dots, \mathcal{A}(\gamma_m)$  are not all integer multiples of a single real number.*

**Remark 1.5.** If  $\lambda$  has infinitely many embedded Reeb orbits, then their symplectic actions can all be integer multiples of a single real number, for example in a prequantization space, or in an ellipsoid  $(\frac{|z_1|^2}{a_1} + \frac{|z_2|^2}{a_2} = 1) \subset \mathbb{C}^2$  with  $a_1/a_2$  rational. Theorem 1.4 (and its proof) does extend to contact forms with infinitely many embedded Reeb orbits if they are isolated in the free loop space.

To state one more result, if  $\lambda$  is a contact form on a closed oriented three-manifold  $Y$ , define the *volume* of  $(Y, \lambda)$  by

$$\text{vol}(Y, \lambda) := \int_Y \lambda \wedge d\lambda. \quad (1.1)$$

One can ask whether there exists a Reeb orbit with an upper bound on the symplectic action in terms of the volume of  $(Y, \lambda)$ . One might also expect that in most cases there are at least three embedded Reeb orbits. The following theorem asserts that assuming another conjecture, at least one of these two statements always holds:

**Theorem 1.6.** *Let  $Y$  be a closed oriented three-manifold, and let  $\lambda$  be a (possibly degenerate) contact form on  $Y$  satisfying the Volume Conjecture, stated below. This holds for example if  $Y = S^3$  and  $\text{Ker}(\lambda)$  is the tight contact structure. Then either:*

- $\lambda$  has at least three embedded Reeb orbits, or
- $\lambda$  has exactly two embedded Reeb orbits, and their symplectic actions  $T, T'$  satisfy  $TT' \leq \text{vol}(Y, \lambda)$ .

## 2 Embedded contact homology and the volume conjecture

To prepare for the proofs Theorem 1.1, 1.4, and 1.6, and to state the volume conjectures that appear in their statements, we need to recall some notions from embedded contact homology (ECH). For more about ECH, see [6] and the references therein.

## 2.1 Definition of embedded contact homology

If  $\lambda$  is nondegenerate, then for each  $\Gamma \in H_1(Y)$  the *embedded contact homology* with  $\mathbb{Z}/2$  coefficients, which we denote by  $ECH_*(Y, \lambda, \Gamma)$ , is defined. (ECH can actually be defined over  $\mathbb{Z}$ , see [11], but  $\mathbb{Z}/2$  coefficients are sufficient for the applications in this paper). This is the homology of a chain complex  $ECC(Y, \lambda, \Gamma, J)$  generated by finite sets  $\alpha = \{(\alpha_i, m_i)\}$  such that each  $\alpha_i$  is a Reeb orbit,  $m_i = 1$  if  $\alpha_i$  is hyperbolic, and

$$\sum_i m_i [\alpha_i] = \Gamma \in H_1(Y).$$

Here a Reeb orbit  $\gamma$  is called *hyperbolic* if the linearized Reeb flow around  $\gamma$  has real eigenvalues. We use the notation  $[\alpha]$  to denote the homology class  $\sum_i m_i [\alpha_i] \in H_1(Y)$ . The  $J$  that enters into the chain complex is an  $\mathbb{R}$ -invariant almost complex structure on  $\mathbb{R} \times Y$  that sends the two-plane field  $\xi$  to itself, rotating it positively with respect to  $d\lambda$ , and satisfies  $J(\partial_s) = R$ , where  $s$  denotes the  $\mathbb{R}$  coordinate on  $\mathbb{R} \times Y$ . The chain complex differential  $\partial$  counts certain mostly embedded  $J$ -holomorphic curves in  $\mathbb{R} \times Y$ . Specifically, if  $\alpha$  and  $\beta$  are two chain complex generators, then the differential coefficient  $\langle \partial\alpha, \beta \rangle \in \mathbb{Z}/2$  is a count of  $J$ -holomorphic curves in  $\mathbb{R} \times Y$ , modulo translation of the  $\mathbb{R}$  coordinate, that are asymptotic as currents to  $\mathbb{R} \times \alpha$  as  $s \rightarrow \infty$  and to  $\mathbb{R} \times \beta$  as  $s \rightarrow -\infty$ . The curves are required to have *ECH index* 1. The ECH index is a certain function of the relative homology class of the curve, explained e.g. in [7]; we do not need to recall the definition here. If  $J$  is generic, then  $\partial$  is well-defined and  $\partial^2 = 0$ , as shown in [10, 11].

The ECH index induces a relative  $\mathbb{Z}/d$ -grading on  $ECH_*(Y, \lambda, \Gamma)$ , where  $d$  denotes the divisibility of  $c_1(\xi) + 2PD(\Gamma)$  in  $H^2(Y)$  mod torsion, see [7, §2.8]. Here  $PD(\Gamma)$  denotes the Poincare dual of  $\Gamma$ .

## 2.2 The isomorphism with Seiberg-Witten Floer cohomology

Although a priori the homology of the chain complex  $ECC(Y, \lambda, \Gamma, J)$  might depend on  $J$ , in fact it does not. This follows from a theorem of Taubes [18, 19, 20, 21] asserting that when  $Y$  is connected, there is a canonical isomorphism between embedded contact homology and a version of Seiberg-Witten Floer cohomology. The precise statement is that there is a canonical isomorphism of relatively graded  $\mathbb{Z}/2$ -modules

$$ECH_*(Y, \lambda, \Gamma) \simeq \widehat{HM}^{-*}(Y, \mathfrak{s}_\xi + PD(\Gamma)), \quad (2.1)$$

where  $\mathfrak{s}_\xi$  is the spin-c structure determined by the oriented two-plane field  $\xi$ , see e.g. [14, Lem. 28.1.1]. (The isomorphism also holds over  $\mathbb{Z}$ .) In particular, there is a well-defined relatively graded  $\mathbb{Z}/2$ -module  $ECH(Y, \xi, \Gamma)$ . By summing over all  $\Gamma \in H_1(Y)$ , one also obtains a well-defined relatively graded  $\mathbb{Z}/2$ -module  $ECH(Y, \xi)$ .

### 2.3 Filtered ECH

If  $\alpha = \{(\alpha_i, m_i)\}$  is a generator of the ECH chain complex, define the *symplectic action* of  $\alpha$  by

$$\mathcal{A}(\alpha) := \sum_i m_i \mathcal{A}(\alpha_i) = \sum_i m_i \int_{\alpha_i} \lambda.$$

It follows from the conditions on  $J$  that the ECH differential decreases the symplectic action. Hence, for any real number  $L$ , one can define the *filtered ECH*, denoted by  $ECH^L(Y, \lambda, \Gamma)$ , to be the homology of the subcomplex of  $ECC$  spanned by generators with action strictly less than  $L$ .

It is shown in [9, Thm. 1.3] that  $ECH^L(Y, \lambda, \Gamma)$  does not depend on the choice of generic  $J$  required to define the chain complex differential. On the other hand,  $ECH^L(Y, \lambda, \Gamma)$ , for fixed  $Y$  and  $\Gamma$ , does depend on the contact form  $\lambda$  and not just on the contact structure  $\xi$ .

As in the previous section, one can remove the homology class  $\Gamma$  from the notation by summing over all possible homology classes. Denote the resulting relatively graded  $\mathbb{Z}/2$  module by  $ECH^L(Y, \lambda)$ .

### 2.4 The U map

If  $Y$  is connected, there is a degree  $-2$  map

$$U : ECH(Y, \lambda, \Gamma) \rightarrow ECH(Y, \lambda, \Gamma). \quad (2.2)$$

It is induced by a chain map  $U_z$  which is defined similarly to the differential  $\partial$ , but instead of counting ECH index 1 curves modulo translation, it counts  $J$ -holomorphic curves of ECH index 2 passing through  $(0, z) \in \mathbb{R} \times Y$ , where  $z$  is a base point in  $Y$  which is not contained in any Reeb orbit, and  $J$  is suitably generic. The connectedness of  $Y$  implies that the induced map (2.2) does not depend on  $z$ . (When  $Y$  is disconnected there is one  $U$  map for each component.) For details see [12, §2.5].

There is an analogous  $U$  map on Seiberg-Witten Floer cohomology, and it is shown in [22, Thm. 1.1] that this agrees with the  $U$  map on ECH under the isomorphism (2.1).

### 2.5 Minimum symplectic action needed to represent a class

Let  $0 \neq \sigma \in ECH(Y, \xi)$ . We now recall from [8] the definition of a real number  $c_\sigma(Y, \lambda)$ , which roughly speaking is the minimum symplectic action needed to represent the class  $\sigma$ .

If  $\lambda$  is nondegenerate, then  $c_\sigma(Y, \lambda)$  is the infimum over  $L$  such that  $\sigma$  is in the image of the inclusion-induced map  $ECH^L(Y, \lambda) \rightarrow ECH(Y, \xi)$ . Note that for any  $J$  as needed to define the chain complex  $ECC(Y, \lambda, J)$ ,

there exists a cycle  $\theta$  in the chain complex representing the class  $\sigma$ , such that every chain complex generator  $\alpha$  that appears in  $\theta$  satisfies  $\mathcal{A}(\alpha) \leq c_\sigma(Y, \lambda)$ , and  $c_\sigma(Y, \lambda)$  is the smallest number with this property. We call a cycle  $\theta$  as above an *action-minimizing representative* of  $\sigma$ .

If  $\lambda$  is degenerate, one defines

$$c_\sigma(Y, \lambda) = \lim_{n \rightarrow \infty} c_\sigma(Y, f_n \lambda), \quad (2.3)$$

where  $f_n : Y \rightarrow \mathbb{R}$  are smooth functions such that  $f_n \lambda$  is nondegenerate and  $\lim_{n \rightarrow \infty} f_n = 1$  in the  $C^0$  topology.

The numbers  $c_\sigma(Y, \lambda)$  then satisfy the following axioms:

(Monotonicity) If  $f : Y \rightarrow \mathbb{R}$  is a smooth function with  $f > 1$ , then  $c_\sigma(Y, \lambda) \leq c_\sigma(Y, f \lambda)$ .

(Scaling) If  $\kappa > 0$  is a constant then  $c_\sigma(Y, \kappa \lambda) = \kappa c_\sigma(Y, \lambda)$ .

(Continuity) If  $f_n : Y \rightarrow \mathbb{R}$  are smooth functions with  $\lim_{n \rightarrow \infty} f_n = 1$  in the  $C^0$  topology, then  $\lim_{n \rightarrow \infty} c_\sigma(Y, f_n \lambda) = c_\sigma(Y, \lambda)$ .

To see that (2.3) is well-defined and to prove the above axioms, one can first show that the Monotonicity and Scaling axioms hold for nondegenerate contact forms, see [8, §4]. It then follows from this that the definition (2.3) does not depend on the sequence  $\{f_n\}$ , and that the Monotonicity, Scaling, and Continuity axioms hold without any nondegeneracy assumption.

## 2.6 The Volume Conjecture

In [8, §8], various conjectures were stated relating the asymptotics of the numbers  $c_\sigma(Y, \lambda)$  to the contact volume (1.1). The most general of these conjectures is as follows. If  $\Gamma \in H_1(Y)$  is such that  $c_1(\xi) + 2PD(\Gamma) \in H^2(Y; \mathbb{Z})$  is torsion, then we know from §2.1 that  $ECC(Y, \xi, \Gamma)$  has a relative  $\mathbb{Z}$ -grading. Choose any normalization of this to an absolute  $\mathbb{Z}$ -grading, and denote the grading of a generator  $x$  by  $I(x) \in \mathbb{Z}$ . We then have:

**Volume Conjecture.** [8, Conj. 8.7] *Let  $(Y, \lambda)$  be a closed connected contact 3-manifold, let  $\Gamma \in H_1(Y)$ , suppose that  $c_1(\xi) + 2PD(\Gamma) \in H^2(Y, \mathbb{Z})$  is torsion, and choose an absolute  $\mathbb{Z}$ -grading as above on  $ECH(Y, \xi, \Gamma)$ . Let  $\{\sigma_k\}_{k=1,2,\dots}$  be a sequence of nonzero elements of  $ECH(Y, \xi, \Gamma)$  with definite gradings satisfying  $\lim_{k \rightarrow \infty} I(\sigma_k) = \infty$ . Then*

$$\lim_{k \rightarrow \infty} \frac{c_{\sigma_k}(Y, \lambda)^2}{I(\sigma_k)} = \text{vol}(Y, \lambda) \quad (2.4)$$

**Remark 2.1.** It follows from [8, Rmk. 3.3, Prop. 4.5] that the Volume Conjecture holds for the boundaries of ellipsoids in  $\mathbb{R}^4$ . It then follows from [8, Prop. 8.6(b)] that the Volume Conjecture holds for all contact forms on  $S^3$  giving the tight contact structure.

Based on the evidence in [8, §8], we expect that the Volume Conjecture holds for all contact 3-manifolds. But to prove the conclusion of Theorem 1.1, we just need the following weaker conjecture:

**Weak Volume Conjecture.** *Let  $(Y, \xi)$  be a closed connected contact 3-manifold. Then there exist nonzero classes  $\{\sigma_k\}_{k \geq 1}$  in  $ECH(Y, \xi)$  such that*

$$U\sigma_{k+1} = \sigma_k \quad (2.5)$$

for all  $k \geq 1$ , and

$$\lim_{k \rightarrow \infty} \frac{c_{\sigma_k}(Y, \lambda)}{k} = 0 \quad (2.6)$$

for every contact form  $\lambda$  with  $\text{Ker}(\lambda) = \xi$ .

**Remark 2.2.** The Volume Conjecture implies the Weak Volume Conjecture. To see this, note that we can always find a class  $\Gamma \in H_1(Y)$  such that  $c_1(\xi) + 2\text{PD}(\Gamma) \in H^2(Y; \mathbb{Z})$  is torsion. It follows from the isomorphism (2.1) of  $ECH(Y, \xi, \Gamma)$  with Seiberg-Witten Floer homology, together with known properties of the latter [14, Lem. 33.3.9, Cor. 35.1.4] that there exists a sequence of  $\{\sigma_k\}_{k \geq 1}$  of nonzero elements of  $ECH(Y, \xi, \Gamma)$  with definite gradings satisfying (2.5). Since the  $U$  map has degree  $-2$ , we have  $I(\sigma_{k+1}) = I(\sigma_k) + 2$ . Hence, the Volume Conjecture applies to give (2.4), which then implies (2.6).

**Remark 2.3.** If  $\{\sigma_k\}_{k \geq 1}$  is a sequence satisfying (2.5), then if  $\{\sigma_k\}_{k \geq 1}$  satisfies (2.6) for one contact form  $\lambda$  with  $\text{Ker}(\lambda) = \xi$ , then it also satisfies (2.6) for every other such contact form, i.e. for the contact form  $f\lambda$  for every smooth function  $f : Y \rightarrow \mathbb{R}^{>0}$ . The reason is that if  $R > \max(f)$ , then it follows from the Monotonicity and Scaling axioms that  $c_{\sigma_k}(Y, f\lambda) < R \cdot c_{\sigma_k}(Y, \lambda)$ .

**Remark 2.4.** It follows from [8, Ex. 8.2] that the Weak Volume Conjecture holds for certain contact forms on  $T^3$  giving the fillable contact structure. It then follows from Remark 2.3 that the Weak Volume Conjecture holds for the fillable contact structure on  $T^3$ . However Theorems 1.1, 1.4, and 1.6 are not so interesting in this case, because here one can use linearized contact homology to prove the existence of infinitely many embedded Reeb orbits, as reviewed in [13, §6.6].

### 3 The key lemma

The key to the proofs of Theorems 1.1, 1.4, and 1.6 is the following:

**Lemma 3.1.** *Let  $Y$  be a closed connected three-manifold and let  $\lambda$  be a (possibly degenerate) contact form on  $Y$  with kernel  $\xi$ . Assume that  $\lambda$  has only finitely many embedded Reeb orbits  $\gamma_1, \dots, \gamma_m$ . Then:*

- (a) *If  $0 \neq \sigma \in ECH(Y, \xi)$ , then  $c_\sigma(Y, \lambda)$  is a nonnegative integer linear combination of  $\mathcal{A}(\gamma_1), \dots, \mathcal{A}(\gamma_m)$ .*
- (b) *If  $\sigma \in ECH(Y, \xi)$  and  $U\sigma \neq 0$ , then  $c_{U\sigma}(Y, \lambda) < c_\sigma(Y, \lambda)$ .*

*Proof.* Fix a nonzero class  $\sigma \in ECH(Y, \xi)$  and write  $L = c_\sigma(Y, \lambda)$ . Choose open tubular neighborhoods  $N_i$  of the Reeb orbits  $\gamma_i$  whose closures are disjoint, and let  $N = \bigcup_{i=1}^m N_i$ . Fix a point  $z \in Y \setminus \overline{N}$  for use in defining the  $U$  map. By shrinking the tubular neighborhoods  $N_i$  if necessary, we may assume that:

- (i) If  $\gamma$  is a Reeb trajectory intersecting both  $z$  and  $\overline{N}$  then  $\int_\gamma \lambda \geq L + 3$ .

Next, choose a sequence of smooth functions  $\{f_n : Y \rightarrow \mathbb{R}^{>0}\}$  such that:

- (ii)  $f_n|_{Y \setminus N} \equiv 1$ ,
- (iii) The contact form  $f_n \lambda$  is nondegenerate,
- (iv)  $\lim_{n \rightarrow \infty} f_n = 1$  in the  $C^1$  topology, and
- (v) Every Reeb orbit of  $f_n \lambda$  with symplectic action less than  $L + 1$  is contained in some  $N_i$ , and has symplectic action within  $1/n$  of an integer multiple of  $\mathcal{A}(\gamma_i)$ .

(The reason we can obtain condition (v) is that otherwise there would be a sequence  $f_n$  such that each  $f_n \lambda$  has a Reeb orbit of action less than  $L + 1$  not contained in  $N$ , or a Reeb orbit in  $N_i$  of action  $< L + 1$  whose action is not within  $\varepsilon$  of an integer multiple of  $\mathcal{A}(\gamma_i)$  for some  $n$ -independent  $\varepsilon > 0$ . Then a subsequence of these Reeb orbits would converge to a Reeb orbit of  $\lambda$  which could not be a multiple of one of the Reeb orbits  $\gamma_i$ .)

It follows from conditions (iii) and (v) that  $c_\sigma(Y, f_n \lambda)$  is within  $m/n$  of an integer linear combination of  $\mathcal{A}(\gamma_1), \dots, \mathcal{A}(\gamma_m)$ . Assertion (a) of the lemma now follows from condition (iv) and the Continuity axiom for  $c_\sigma$ .

To prove (b), continue to fix the above data, and assume that  $U\sigma \neq 0$ . For each  $n$ , choose a generic almost complex structure  $J_n$  on  $\mathbb{R} \times Y$  as needed to define the filtered ECH chain complex  $ECC^{L+1}(Y, f_n \lambda, J_n)$  and the chain map  $U_z$  on it. Specifically, we need  $J_n$  to satisfy the genericity conditions listed in the first paragraph of [11, §10], for  $J_n$ -holomorphic curves counted



by  $\partial$  or  $U_z$  whose positive ends have total action less than  $L + 1$ . These conditions on  $J_n$  can all be achieved by perturbing  $J_n$  near the Reeb orbits of action less than  $L + 1$ . So by condition (v) above, we can arrange that the almost complex structures  $J_n$  agree with a fixed almost complex structure  $J_0$  on  $\mathbb{R} \times (Y \setminus N)$ .

We know from the proof of (a) that if  $n$  is sufficiently large then  $c_\sigma(Y, f_n\lambda) < L + 1$ , so we can choose an action-minimizing representative  $\theta_n$  of  $\sigma$  in  $ECC^{L+1}(Y, f_n\lambda)$ .

*Claim.* There exists  $\delta > 0$  and a positive integer  $n_0$  such that if  $n \geq n_0$  and  $C_n$  is a  $J_n$ -holomorphic curve counted by  $U_z\theta_n$ , then  $\int_{C_n} d(f_n\lambda) \geq \delta$ .

The Claim implies (b), because it implies that if  $n \geq n_0$  then  $c_{U\sigma}(Y, f_n\lambda) \leq c_\sigma(Y, f_n\lambda) - \delta$ , and so by the Continuity axiom  $c_{U\sigma}(Y, \lambda) \leq c_\sigma(Y, \lambda) - \delta$ .

*Proof of Claim:* Recall that the conditions on  $J_n$  imply that if  $C_n$  is any  $J_n$ -holomorphic curve, then  $d(f_n\lambda)$  is pointwise nonnegative on  $C_n$ , with equality only where the tangent space to  $C_n$  is the span of the  $\mathbb{R}$  direction and the Reeb direction (or where  $C_n$  is singular, but this does not happen for curves counted by  $U_z\theta_n$ ). In particular,  $\int_{C_n} d(f_n\lambda) \geq 0$ . Consequently, if the Claim is false, then we can find an increasing sequence  $\{n_i\}_{i \geq 1}$  of positive integers, and for each  $i$  a  $J_{n_i}$ -holomorphic curve  $C_{n_i}$  counted by  $U_z\theta_{n_i}$ , such that  $\lim_{i \rightarrow \infty} \int_{C_{n_i}} d(f_{n_i}\lambda) = 0$ .

We now use the following proposition, which is a special case of a result of Taubes [16, Prop. 3.3]:

**Proposition 3.2.** *Let  $(X, \omega)$  be a compact symplectic 4-manifold with boundary with a compatible almost complex structure  $J$ . Let  $\{C_i\}_{i \in \mathbb{N}}$  be a sequence of compact  $J$ -holomorphic curves in  $X$  with boundary contained in  $\partial X$ , and suppose that there exists  $E > 0$  such that  $\int_{C_i} \omega < E$  for all  $i$ . Then one can pass to a subsequence such that:*

*(Convergence as currents) The curves  $\{C_i\}$  converge weakly as currents to a compact  $J$ -holomorphic curve  $C_0$  with boundary in  $\partial X$  such that  $\int_{C_0} \omega \leq E$ , and*

*(Pointwise convergence)*

$$\lim_{i \rightarrow \infty} \left( \sup_{x \in C_{i^*}} \text{dist}(x, C_0) + \sup_{x \in C_0} \text{dist}(x, C_{i^*}) \right) = 0.$$

We apply the above proposition to the intersections of the holomorphic curves  $C_{n_i}$  with  $X = [-1, 1] \times (Y \setminus N)$ , with the symplectic form  $\omega = d(e^s\lambda)$ . To see why we have the necessary upper bound on  $\omega$  to apply the proposition, given  $i$ , choose  $s_+ \in [1, 2]$  and  $s_- \in [-2, -1]$  such that  $C_{n_i}$  is tranverse to

$\{s_{\pm}\} \cap Y$ . Then since  $d(e^s f_{n_i} \lambda)$  and  $d(f_{n_i} \lambda)$  are pointwise nonnegative on  $C_{n_i}$ , we have an upper bound

$$\begin{aligned} \int_{C_{n_i} \cap ([-1, 1] \times (Y \setminus N))} \omega &\leq \int_{C_{n_i} \cap ([s_-, s_+] \times Y)} d(e^s f_{n_i} \lambda) \\ &= e^{s_+} \int_{C_{n_i} \cap (\{k\} \times Y)} f_{n_i} \lambda - e^{s_-} \int_{C_{n_i} \cap (\{-k\} \times Y)} f_{n_i} \lambda \\ &< e^2(L + 1). \end{aligned}$$

So we can pass to a subsequence such that  $C_{n_i} \cap ([-1, 1] \times (Y \setminus N))$  converges in the sense of Proposition 3.2 to a (possibly multiply covered)  $J_0$ -holomorphic curve  $C_0$  in  $[-1, 1] \times (Y \setminus N)$ . By the “pointwise convergence” condition, the curve  $C_0$  contains the point  $(0, z)$ , since each  $C_{n_i}$  does.

Since  $C_0$  is  $J_0$ -holomorphic, it follows that  $d\lambda$  is pointwise nonnegative on  $C_0$ , with equality only where  $C_0$  is singular or the tangent space of  $C_0$  is the span of the  $\mathbb{R}$  direction and the Reeb direction. In particular,

$$\int_{C_0} d\lambda \geq 0. \quad (3.1)$$

In fact, the inequality (3.1) must be strict. Otherwise  $C_0$ , regarded as a current, is invariant under translation of the  $[-1, 1]$  coordinate on  $[-1, 1] \times (Y \setminus N)$ . It follows that  $C_0 \cap (\{0\} \times (Y \setminus N))$  is tangent to the Reeb vector field for  $\lambda$ . In particular,  $C_0 \cap (\{0\} \times (Y \setminus N))$ , regarded as a subset of  $Y$ , contains a Reeb trajectory for  $\lambda$  passing through  $z$  with endpoints on  $\partial \overline{N}$ . So by (i) above,

$$\int_{C_0 \cap (\{0\} \times (Y \setminus N))} \lambda \geq L + 3.$$

By the convergence of currents above, it follows that

$$\int_{C_{n_i} \cap (\{s\} \times (Y \setminus N))} f_{n_i} \lambda \geq L + 2 \quad (3.2)$$

whenever  $i$  is sufficiently large and  $s \in [-1, 1]$  is such that  $C_{n_i}$  is transverse to  $\{s\} \times Y$ . When this transversality holds, we orient  $C_{n_i} \cap (\{s\} \times Y)$ , regarded as a submanifold, by the “ $\mathbb{R}$ -direction first” convention. The conditions on  $J_{n_i}$  imply that  $f_{n_i} \lambda$  is pointwise nonnegative on this oriented one-manifold, so it follows from (3.2) that

$$\int_{C_{n_i} \cap (\{s\} \times Y)} f_{n_i} \lambda \geq L + 2. \quad (3.3)$$

But this is impossible, because the left hand side of (3.3) must be less than or equal to the maximum symplectic action of a generator in  $\theta_{n_i}$ , which is less than  $L + 1$ . This contradiction proves that the inequality (3.1) is strict.

Given this, let  $\delta = \frac{1}{2} \int_{C_0} d\lambda > 0$ . It follows from the convergence of currents that if  $i$  is sufficiently large then

$$\begin{aligned} \int_{C_{n_i}} d(f_{n_i}\lambda) &\geq \int_{C_{n_i} \cap ([-1,1] \times (Y \setminus N))} d(f_{n_i}\lambda) \\ &= \int_{C_{n_i} \cap ([-1,1] \times (Y \setminus N))} d\lambda \\ &\geq \int_{C_0} d\lambda - \delta \\ &= \delta. \end{aligned}$$

This contradicts our assumption that  $\lim_{i \rightarrow \infty} \int_{C_{n_i}} d(f_{n_i}\lambda) = 0$  and thus completes the proof of the Claim, and with it Lemma 3.1.  $\square$

**Remark 3.3.** In the above argument we can not quote the SFT compactness theorem from [1], because that result assumes both a genus bound (which one does not have in ECH) as well as nondegeneracy of the contact form. This is why we use Taubes's approach via currents. Although this is only applicable in four dimensions, if one has a genus bound then one can cite [3] for similar arguments in higher dimensions.

## 4 Proofs of theorems

*Proof of Theorem 1.1.* This follows from Theorem 1.4.  $\square$

*Proof of Theorem 1.4.* Suppose that  $\lambda$  has only finitely many embedded Reeb orbits  $\gamma_1, \dots, \gamma_m$ , and suppose that their symplectic actions  $\mathcal{A}(\gamma_1), \dots, \mathcal{A}(\gamma_m)$  are all integer multiples of a single real number  $T > 0$ . Let  $\{\sigma_k\}_{k \geq 1}$  be any sequence satisfying (2.5). Then by Lemma 3.1, we have  $c_{\sigma_k}(Y, \lambda) = n_k T$  where  $\{n_k\}_{k \geq 1}$  is a strictly increasing sequence of nonnegative integers. It follows that

$$\liminf_{k \rightarrow \infty} \frac{c_{\sigma_k}(Y, \lambda)}{k} \geq T, \quad (4.1)$$

so that (2.6) cannot hold. This contradicts the Weak Volume Conjecture.  $\square$

*Proof of Theorem 1.6.* Suppose there are fewer than three embedded Reeb orbits. We know from Theorem 1.1 that there are exactly two embedded Reeb orbits; denote their symplectic actions by  $T$  and  $T'$ .

Let  $\{\sigma_k\}_{k \geq 1}$  be a sequence as provided by Remark 2.2. By Lemma 3.1, we have  $c_{\sigma_k}(Y, \lambda) = n_k T + n'_k T'$  where  $n_k$  and  $n'_k$  are nonnegative integers such that  $n_{k+1} T + n'_{k+1} T' > n_k T + n'_k T'$ . It follows from this that

$$\lim_{k \rightarrow \infty} \frac{c_{\sigma_k}(Y, \lambda)^2}{k} \geq 2TT'. \quad (4.2)$$

To see this, note that if we fix  $k$  and write  $L = c_{\sigma_k}(Y, \lambda) = n_k T + n_{k'} T'$ , then  $k$  is less than or equal to the number of pairs of nonnegative integers  $(x, y)$  with  $xT + yT' \leq L$ , which is the number of lattice points in the triangle enclosed by the line  $Tx + T'y = L$  and the  $x$  and  $y$  axes, which is  $L^2/(2TT') + O(L)$ , compare [8, §3.3]. On the other hand, since the  $U$  map has degree  $-2$ , we have

$$\lim_{k \rightarrow \infty} \frac{I(\sigma_k)}{k} = 2. \quad (4.3)$$

Putting (4.2) and (4.3) into (2.4) gives  $\text{vol}(Y, \lambda) \geq TT'$ .  $\square$

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